SPRING 2025 MATH 540: QUIZ 1

Name:

1. State the Well Ordering Principle (3 points)

Solution. Every non-empty set of natural numbers has a least element.

2. Prove that if $a, r \in \mathbb{R}$ and $r \neq 1$, then for all $n \geq 1$, $a + ar + \cdots + ar^n = \frac{a(r^{n+1}-1)}{r-1}$. (3 points)

Solution. The proof is by induction on n. Suppose $n = 1$. Then the left hand side of the given equation is $a + ar$. The right hand side of the equation is

$$\frac{a(r^{1+1}-1)}{r-1} = \frac{a(r^2-1)}{r-1} = a(r+1) = a + ar,$$

which is what we want.

Assume the result is true for $n-1$, with $n > 1$. Then we have

$$a + ar + \cdots + ar^{n-1} = \frac{a(r^{n-1+1}-1)}{r-1}.$$

Adding ar^n to both sides of the equation gives

$$\begin{aligned}
a + ar + \cdots + ar^{n-1} + ar^n &= \frac{a(r^{n-1+1}-1)}{r-1} + ar^n \\
a + ar + \cdots + ar^{n-1} + ar^n &= \frac{a(r^{n-1+1}-1)}{r-1} + \frac{ar^n(r-1)}{r-1} \\
a + ar + \cdots + ar^{n-1} + ar^n &= \frac{ar^n - a + ar^{n+1} - ar^n}{r-1} \\
a + ar + \cdots + ar^{n-1} + ar^n &= \frac{a(r^{n+1}-1)}{r-1}
\end{aligned}$$

3. Use the Euclidean algorithm to find the GCD of 216 and 135. Then use backwards substitution to write the GCD as an integer combination of 216 and 135, as expected, via Bezout's Principle. (4 points)

Solution. Using Euclidean algorithm, we have:

$$\begin{aligned}216 &= 135 \cdot 1 + 81\\ 135 &= 81 \cdot 1 + 54\\ 81 &= 1 \cdot 54 + 27\\ 54 &= 27 \cdot 2 + 0,\end{aligned}$$

showing that $27 = \mathrm{GCD}(216, 135)$. Working with these equations in reverse order, we have

$$\begin{aligned}27 &= -1 \cdot 54 + 81\\ 27 &= -1 \cdot (135 - 1 \cdot 81) + 81\\ 27 &= -1 \cdot 135 + 2 \cdot 81\\ 27 &= -1 \cdot 135 + 2 \cdot (216 - 1 \cdot 135)\\ 27 &= -3 \cdot 135 + 2 \cdot 216.\end{aligned}$$